

# Inflexible $CR$ submanifolds

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## Abstract

In this paper we introduce the concept of *inflexible*  $CR$  submanifolds. These are  $CR$  submanifolds of some complex Euclidean space such that any compactly supported  $CR$  deformation is again globally  $CR$  embeddable into some complex Euclidean space. Our main result is that any 2-pseudoconcave quadratic  $CR$  submanifold of type  $(n, d)$  in  $\mathbb{C}^{n+d}$  is inflexible.

## 1 Introduction

In this paper, we shall be interested in proving embedding results for compactly supported perturbations of embedded  $CR$  manifolds.

Here an abstract  $CR$  manifold of type  $(n, d)$  is a triple  $(M, HM, J)$ , where  $M$  is a smooth real manifold of dimension  $2n + d$ ,  $HM$  is a subbundle of rank  $2n$  of the tangent bundle  $TM$ , and  $J : HM \rightarrow HM$  is a smooth fiber preserving bundle isomorphism with  $J^2 = -\text{Id}$ . We also require that  $J$  be formally integrable; i.e. that we have

$$[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$$

where

$$T^{0,1}M = \{X + iJX \mid X \in \Gamma(M, HM)\} \subset \Gamma(M, \mathbb{C}TM),$$

with  $\Gamma$  denoting smooth sections.

The  $CR$  dimension of  $M$  is  $n \geq 1$  and the  $CR$  codimension is  $d \geq 1$ .

A problem of great interest is to decide which  $CR$  manifolds  $M$  admit  $CR$  embeddings into some complex Euclidean space. Namely, can one find a smooth embedding  $\varphi$  of  $M$  into  $\mathbb{C}^N$  such that the induced  $CR$  structure

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$\varphi_*(T^{0,1}M)$  on  $\varphi(M)$  coincides with the  $CR$  structure  $T^{0,1}(\mathbb{C}^N) \cap CT(\varphi(M))$  from the ambient space  $\mathbb{C}^N$ .

Typically, examples of non-embeddable  $CR$  structures arise as deformations of  $CR$  submanifolds of some complex Euclidean space. For example, Rossi [R] constructed small real analytic deformations of the standard  $CR$  structure on the 3-sphere  $S^3$  in  $\mathbb{C}^2$ , and the resulting abstract  $CR$  structures fail to  $CR$  embed globally into  $\mathbb{C}^2$ . Also Nirenberg's famous local nonembeddability examples [Ni] can be interpreted as small (local) deformations of the Heisenberg structure on  $\mathbb{H}^2 \subset \mathbb{C}^2$ . The examples by Nirenberg were later on extended to higher dimensions by Jacobowitz and Trèves [JT].

However, there is something special about Nirenberg's three-dimensional examples: Since the formal integrability condition is always satisfied in this situation, one can easily modify the examples to obtain small (global) deformations of the Heisenberg structure  $\mathbb{H}^2$ . Moreover, these deformations are compactly supported (in the sense that the deformations coincide with the given Heisenberg structure outside a compact set). For the examples of Jacobowitz and Trèves, it is not clear if this is possible.

In fact, as soon as the  $CR$  dimension is greater than one, the integrability conditions come into play, and they make it much more difficult to construct deformations. However, when  $M$  is given as a  $CR$  submanifold of some complex Euclidean space, one can always obtain compact deformations of the  $CR$  structure on  $M$  by making a small compact geometric deformation of  $M$  within the complex Euclidean space. We refer to this as "punching  $M$ ". But it is not clear if there exists other compact deformations of the abstract  $CR$  structure on  $M$ , which render  $M$  no longer embeddable as a  $CR$  submanifold of the complex Euclidean space, such as in Nirenberg's example.

Therefore in the present paper, we want to discuss the following problem: Suppose  $f : (M, HM, J) \longrightarrow \mathbb{C}^{n+k}$  is a  $CR$  embedding, and  $(M', HM', J')$  is small, compactly supported  $CR$  deformation of  $(M, HM, J)$ . Does it follow that it also admits a  $CR$  embedding  $f'$  with  $f'$  close to  $f$ ?

An answer to this question clearly depends on the Levi-form of  $M$ , so let us now recall its intrinsic definition.

We denote by  $H^oM = \{\xi \in T^*M \mid \langle X, \xi \rangle = 0, \forall X \in H_{\pi(\xi)}M\}$  the *characteristic conormal bundle* of  $M$ . Here  $\pi : TM \longrightarrow M$  is the natural projection. To each  $\xi \in H_p^oM \setminus \{0\}$ , we associate the Levi form at  $\xi$  :

$$\mathcal{L}_p(\xi, X) = \xi([J\tilde{X}, \tilde{X}]) = d\tilde{\xi}(X, JX) \text{ for } X \in H_pM$$

which is Hermitian for the complex structure of  $H_p M$  defined by  $J$ . Here  $\tilde{\xi}$  is a section of  $H^o M$  extending  $\xi$  and  $\tilde{X}$  a section of  $HM$  extending  $X$ .

Following [HN]  $M$  is called  $q$ -pseudoconcave,  $0 \leq q \leq \frac{n}{2}$  if for every  $p \in M$  and every characteristic conormal direction  $\xi \in H_p^o M \setminus \{0\}$ , the Levi form  $\mathcal{L}_p(\xi, \cdot)$  has at least  $q$  negative and  $q$  positive eigenvalues.

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## 2 Definitions and statement of the main results

Let  $(M, HM, J)$  be  $CR$  manifold of type  $(n, d)$  globally  $CR$  embedded into some complex Euclidean space. We say that  $(M, HM, J)$  admits a *compactly supported  $CR$  deformation* if there exists a family  $(M_a, HM_a, J_a)_{a>0}$  of abstract  $CR$  manifolds depending smoothly on a real parameter  $a > 0$  and converging to  $(M, HM, J)$  as  $a$  tends to 0 in the usual  $\mathcal{C}^\infty$  topology; we also require that  $(M_a, HM_a, J_a) = (M, HM, J)$  for every  $a > 0$  outside some compact  $K$  of  $M$  not depending on  $a$ .

We say that  $(M, HM, J)$  is a *flexible  $CR$  submanifold* if it admits a compactly supported  $CR$  deformation  $(M_a, HM_a, J_a)_{a>0}$  such that for every sufficiently small  $a > 0$ , the  $CR$  structure  $(M_a, HM_a, J_a)$  is not globally  $CR$  embeddable into some complex Euclidean space. So, for example, the Heisenberg  $CR$  structure  $\mathbb{H}^2$  in  $\mathbb{C}^2$  is flexible.

We say that  $(M, HM, J)$  is an *inflexible  $CR$  submanifold* if it is not flexible. That means that  $(M, HM, J)$  is inflexible if and only if for every compactly supported  $CR$  deformation  $(M_a, HM_a, J_a)_{a>0}$  of  $(M, HM, J)$ , the  $CR$  manifold  $(M_a, HM_a, J_a)$  is globally  $CR$  embeddable into some complex Euclidean space.

In other words, a flexible  $CR$  submanifold admits a compactly supported  $CR$  deformation that "pops out" of the space of globally  $CR$  embeddable manifolds. On the other hand, for an inflexible  $CR$  submanifold, any compactly supported  $CR$  deformation stays in the space of globally  $CR$  embeddable manifolds.

*Remark:* In the definitions above, we also allow compact deformations which are only defined for a sequence of  $a$ 's tending to zero.

Our main result is as follows:

**Theorem 2.1**

Let  $M$  be a quadratic  $CR$  submanifold of type  $(n, d)$  in  $\mathbb{C}^{n+d}$  that is 2-pseudoconcave. Let  $(M_a, HM_a, J_a)_{a>0}$  be a compactly supported  $CR$  deformation of  $(M, HM, J)$ . Then, given any smooth  $CR$  function  $f : (M, HM, J) \rightarrow \mathbb{C}$ , there is a  $CR$  function  $f_a : (M_a, HM_a, J_a) \rightarrow \mathbb{C}$  as close to  $f$  as we please, provided  $a$  is sufficiently close to 0.

Moreover,  $f_a$  can be chosen to coincide with the given  $f$  outside a compact of  $M$ . In particular,  $(M_a, HM_a, J_a)$  is  $CR$  embeddable into  $\mathbb{C}^{n+d}$  for a sufficiently close to 0.

Here a *quadratic  $CR$  submanifold* is a submanifold of  $\mathbb{C}^{n+d}$  of the form

$$M = \{z \in \mathbb{C}^{n+d} \mid \operatorname{Im} z_\ell = H_\ell(z_1, \dots, z_n), \ n+1 \leq \ell \leq n+d\},$$

where the  $H_\ell$ 's are quadratic hermitian forms on  $\mathbb{C}^n$ .

" $f_a$  as close to  $f$  as we please" means that for any given  $\ell \in \mathbb{N}$ , any given compact  $K$  of  $M$  and arbitrary small  $\varepsilon > 0$ , one can find a  $CR$  function  $f_a : (M_a, HM_a, J_a) \rightarrow \mathbb{C}$  such that the  $\mathcal{C}^\ell$  norm of  $f - f_a$  on  $K$  is less than  $\varepsilon$ .

In particular, Theorem 2.1 implies that for a 2-pseudoconcave quadratic  $CR$  submanifold, any compactly supported  $CR$  deformation amounts to "punching  $M$ ": any of the ambient complex coordinate functions is a  $CR$  function on  $M$ . Our theorem yields that we can make arbitrarily small modifications of these coordinate functions inside a compact subset of  $M$  to obtain global  $CR$  coordinate functions on the deformed  $CR$  manifolds.

The last statement of Theorem 2.1 combined with the definition of "inflexible" immediately gives the following

**Corollary 2.2**

Let  $M$  be 2-pseudoconcave quadratic  $CR$  submanifold of type  $(n, d)$  in  $\mathbb{C}^{n+d}$ . Then  $M$  is inflexible.

### 3 A first example

The idea of the proof of Theorem 2.1 is as follows: For a given  $CR$  function  $f$  on  $M$  we want to find a  $CR$  function on  $M_a$  which is very close to the given  $f$ . Therefore we want to solve the Cauchy-Riemann equations  $\bar{\partial}_{M_a} u = \bar{\partial}_{M_a} f$  with  $u$  having compact support and the  $\mathcal{C}^k$ -norms of  $u$  being controlled by some  $\mathcal{C}^l$ -norms of  $\bar{\partial}_{M_a}$  (uniformly with respect to  $a$ ).

In this section, we will explicitly carry out the proof of our main result 2.1 in all details for the easiest example of a 2-pseudoconcave  $CR$  manifold. Namely let  $M \subset \mathbb{C}^5$  be the real hypersurface defined by

$$M = \{(z_1, z_2, z_3, z_4, x + iy) \mid y = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2\}. \quad (3.1)$$

Then  $M$  is a 2-pseudoconcave  $CR$  manifold of type  $(4, 1)$ . To abbreviate notations, we also define  $z = (z_1, z_2, z_3, z_4)$  and  $|z|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2$ . A straightforward computation shows that  $T^{0,1}M$  is spanned by

$$\bar{L}_j = \frac{\partial}{\partial \bar{z}_j} - i\epsilon_j z_j \frac{\partial}{\partial x}, \quad j = 1, 2, 3, 4,$$

where  $\epsilon_1 = \epsilon_2 = -1$  and  $\epsilon_3 = \epsilon_4 = 1$ . For  $u = \sum_1^4 u_j d\bar{z}_j \in \mathcal{C}_{0,1}^\infty(M)$  we have

$$\bar{\partial}_M u = \sum_{j,k=1}^4 \bar{L}_k(u_j) d\bar{z}_k \wedge d\bar{z}_j.$$

Next, we consider the volume element

$$dV = \left(\frac{i}{2}\right)^4 e^{|z|^2} \bigwedge_{j=1}^4 dz_j \wedge d\bar{z}_j \wedge dx = \frac{1}{16} e^{|z|^2} \bigwedge_{j=1}^4 dz_j \wedge d\bar{z}_j \wedge dx$$

on  $M$ , and we denote by  $\|\cdot\|$  the  $L^2$ -norm of  $(0, q)$ -forms on  $M$  with respect to this volume element, where the pointwise norms of  $(0, q)$ -forms on  $M$  is the one induced by the standard euclidean metric on  $\mathbb{C}^5$ . The corresponding  $L^2$ -spaces will be denoted by  $L_{0,q}^2(M, |z|^2)$ . Then  $\bar{\partial}_M^*$ , the formal adjoint of  $\bar{\partial}_M$  with respect to  $\|\cdot\|$  can be computed as follows:

$$\bar{\partial}_M^* u = -e^{-|z|^2} \sum_{j=1}^4 L_j(u_j e^{|z|^2})$$

for  $u \in \mathcal{D}^{0,1}(M)$ .

First we will prove the following  $L^2$  estimates on  $M$ :

**Lemma 3.1**

Let  $M$  be defined as in (3.1).

1. For all  $u \in L_{0,1}^2(M, |z|^2) \cap \text{Dom}(\bar{\partial}_M) \cap \text{Dom}(\bar{\partial}_M^*)$  we have

$$2\|u\|^2 \leq \|\bar{\partial}_M u\|^2 + \|\bar{\partial}_M^* u\|^2. \quad (3.2)$$

2. For all  $u \in L^2_{0,0}(M, |z|^2) \cap \text{Dom}(\bar{\partial}_M)$  we have

$$4\|u\|^2 \leq \|\bar{\partial}_M u\|^2 \quad (3.3)$$

*Proof.* Throughout the proof of this Lemma, we identify  $M$  with  $\mathbb{C}^4 \times \mathbb{R}$ . We will begin by showing how to reduce the proof of (3.2) to an estimate for an easier differential operator. Therefore we introduce the partial Fourier transform with respect to the variable  $x$ :

$$\tilde{u}(z, \xi) = \int e^{-i\langle x, \xi \rangle} u(z, x) dx$$

(for differential forms, this partial Fourier transform is defined component-wise).

Now an easy computation shows that for  $u \in \mathcal{D}^{0,1}(M)$  we have

$$\begin{aligned} \widetilde{\bar{\partial}_M u}(z, \xi) &= \sum_{j,k=1}^4 \widetilde{\bar{L}_k(u_j)}(z, \xi) d\bar{z}_k \wedge d\bar{z}_j \\ &= \sum_{j,k=1}^4 \left( \frac{\partial}{\partial \bar{z}_k} u_j - i\epsilon_k z_k \frac{\partial}{\partial x} u_j \right) \sim(z, \xi) d\bar{z}_k \wedge d\bar{z}_j \\ &= \sum_{j,k=1}^4 \left( \frac{\partial}{\partial \bar{z}_k} \tilde{u}_j(z, \xi) + \epsilon_k z_k \xi \tilde{u}_j(z, \xi) \right) d\bar{z}_k \wedge d\bar{z}_j \\ &= \bar{\partial}_{(z)} \tilde{u}(z, \xi), \end{aligned}$$

where  $\bar{\partial}_{(z)}$  is defined by

$$\bar{\partial}_{(z)} v(z, \xi) = \sum_{j,k=1}^4 \bar{\partial}_k v_j d\bar{z}_k \wedge d\bar{z}_j.$$

Here  $\bar{\partial}_k v_j = \frac{\partial}{\partial \bar{z}_k} v_j + \epsilon_k z_k \xi v_j$  is of order 0 in  $\xi$ . Similarly, we get

$$\begin{aligned} \widetilde{\bar{\partial}_M^* u}(z, \xi) &= -\left( \sum_{j=1}^4 L_j \widetilde{u_j} + \bar{z}_j u_j \right)(z, \xi) \\ &= \delta_{(z)} \tilde{u}(z, \xi), \end{aligned}$$

where

$$\delta_{(z)} v(z, \xi) = \sum_{j=1}^4 (\delta_j v_j)(z, \xi)$$

with  $\delta_j v_j = -\frac{\partial}{\partial z_j} v_j + \epsilon_j \bar{z}_j \xi v_j - \bar{z}_j v_j$ . Note that also  $\delta_j$  is of order 0 in  $\xi$ .

Now, as in [H1] we compute

$$\begin{aligned}
|\bar{\partial}_{(z)}v|^2 &= \left| \sum_{j,k=1}^4 \bar{\partial}_k v_j d\bar{z}_k \wedge dz_j \right|^2 \\
&= \frac{1}{2} \sum_{j,k=1}^4 |\bar{\partial}_k v_j - \bar{\partial}_j v_k|^2 \\
&= \sum_{j,k=1}^4 |\bar{\partial}_j v_k|^2 - \sum_{j,k=1}^4 \bar{\partial}_k v_j \overline{\bar{\partial}_j v_k}
\end{aligned} \tag{3.4}$$

Also we have

$$\begin{aligned}
|\delta_{(z)}v|^2 &= \left| \sum_{j=1}^4 \delta_j v_j \right|^2 = \sum_{j,k=1}^4 \delta_j v_j \overline{\delta_k v_k} \\
&= \sum_{j=1}^4 |\delta_j v_j|^2 + \sum_{j \neq k} \delta_j v_j \overline{\delta_k v_k}
\end{aligned} \tag{3.5}$$

Summing up (3.4) and (3.5) we obtain

$$\begin{aligned}
\int_{\mathbb{C}^4} (|\bar{\partial}_{(z)}v|^2 + |\delta_{(z)}v|^2) \exp(|z|^2) \left(\frac{i}{2}\right)^4 \bigwedge_{j=1}^4 dz_j \wedge d\bar{z}_j = \\
\sum_{j=1}^4 \|\delta_j v_j\|_z^2 + \sum_{j \neq k} \|\bar{\partial}_j v_k\|_z^2 + \sum_{j \neq k} \ll [\bar{\partial}_k, \delta_j] v_j, v_k \gg_z .
\end{aligned}$$

Here we have used that  $\bar{\partial}_k$  and  $\delta_k$  are adjoint operators. To abbreviate notations, we have introduced  $\|\cdot\|_z$  to denote partial integration with respect to the  $z = (z_1, z_2, z_3, z_4)$  variables:

$$\|v\|_z^2 = \int_{z \in \mathbb{C}^4} |v(z, \xi)|^2 \exp(|z|^2) \left(\frac{i}{2}\right)^4 \bigwedge_{j=1}^4 dz_j \wedge d\bar{z}_j.$$

Since  $[\bar{\partial}_k, \delta_j] = 0$  for  $j \neq k$  we obtain

$$\|\bar{\partial}_{(z)}v\|_z^2 + \|\delta_{(z)}v\|_z^2 = \sum_{j \neq k} \|\bar{\partial}_j v_k\|_z^2 + \sum_{j=1}^4 \|\delta_j v_j\|_z^2 \tag{3.6}$$

Also, a straightforward computation shows that

$$[\bar{\partial}_j, \delta_j] = -1 + 2\epsilon_j \xi \tag{3.7}$$

This will be used to show that for each fixed  $k \in \{1, 2, 3, 4\}$  we have

$$\sum_{\substack{j=1 \\ j \neq k}}^4 \|\bar{\partial}_j v_k\|_z^2 \geq 2\|v_k\|_z^2 \quad (3.8)$$

Assume e.g.  $k=4$ . From (3.7) we then obtain

$$\|\delta_j v_4\|_z^2 - \|\bar{\partial}_j v_4\|_z^2 = (-1 + 2\epsilon_j \xi) \|v_4\|_z^2. \quad (3.9)$$

It follows that

$$\begin{aligned} \sum_{j=1}^3 \|\bar{\partial}_j v_4\|_z^2 &\geq \sum_{j=1,3} \|\bar{\partial}_j v_4\|_z^2 \\ &= \sum_{j=1,3} \|\delta_j v_4\|_z^2 + \sum_{j=1,3} (1 - 2\epsilon_j \xi) \|v_4\|_z^2 \\ &\geq 2\|v_4\|_z^2 - 2\xi(-1 + 1)\|v_4\|_z^2 \\ &= 2\|v_4\|_z^2, \end{aligned}$$

which proves (3.8) for  $k = 4$ . The remaining cases are similar.

Combining (3.6) and (3.8) we have proved that

$$2\|v\|_z^2(\xi) \leq \|\bar{\partial}_{(z)} v\|_z^2(\xi) + \|\delta_{(z)} v\|_z^2(\xi)$$

for every fixed  $\xi \in \mathbb{R}$ . Setting  $v = \tilde{u}$  and integrating this inequality with respect to  $\xi$  we obtain from the definition of the operators  $\bar{\partial}_{(z)}$  and  $\delta_{(z)}$

$$2\|\tilde{u}\|^2 \leq \|\widetilde{\bar{\partial}_M u}\|^2 + \|\widetilde{\bar{\partial}_M^* u}\|^2$$

for all  $u \in \mathcal{D}^{0,1}(M)$ .

The Plancherel theorem permits to conclude that

$$2 \int_M |u|^2 dV \leq \int_M (|\bar{\partial}_M u|^2 + |\bar{\partial}_M^* u|^2) dV$$

for  $u \in \mathcal{D}^{0,1}(M)$ . Obviously, the restriction of the standard euclidean metric to  $M$  is complete, therefore the above estimate extends to all  $u \in L_{0,1}^2(M, |z|^2) \cap \text{Dom}(\bar{\partial}_M) \cap \text{Dom}(\bar{\partial}_M^*)$ , which proves the first statement of the Lemma.

The proof of (3.3) is similar. Indeed, using the partial Fourier transform, the proof of (3.3) is again reduced to the estimate of  $\sum_{j=1}^4 \|\bar{\partial}_j v\|_z^2$ , where  $\bar{\partial}_j$



is defined as before. But using (3.9) we get

$$\begin{aligned} \sum_{j=1}^4 \|\bar{\partial}_j v\|_z^2 &= \sum_{j=1}^4 \|\delta_j v\|_z^2 + \sum_{j=1}^4 (1 - 2\epsilon_j \xi) \|v\|_z^2 \\ &\geq 4\|v\|_z^2 - 2\xi(-1 - 1 + 1 + 1)\|v\|_z^2 \\ &= 4\|v\|_z^2, \end{aligned}$$

This completes the proof of the Lemma by the same arguments as before.  $\square$

Next, we use again that  $M$  is 2-pseudoconcave (this condition is clearly stable under small perturbations). This implies that we have a uniform subelliptic estimate in degree  $(0, 1)$  (see [FK]):

For every compact  $K$  of  $M$ , there exists a constant  $C_K > 0$  independent of  $a$  such that

$$\|u\|_{\frac{1}{2}}^2 \leq C_K (\|\bar{\partial}_{M_a} u\|^2 + \|\bar{\partial}_{M_a}^* u\|^2 + \|u\|^2) \quad (3.10)$$

for all  $u \in \mathcal{D}_K^{0,1}(M_a)$ .

Combining Lemma 3.1 and (3.10), we can establish an  $L^2$  a priori estimate in degree  $(0, 1)$ , which is uniform with respect to  $a$  (in the sense that the constant involved does not depend on  $a$ ).

### Lemma 3.2

There is  $a_0 > 0$  and a constant  $C > 0$  such that

$$\|u\|^2 \leq C (\|\bar{\partial}_{M_a} u\|^2 + \|\bar{\partial}_{M_a}^* u\|^2)$$

for all  $u \in L_{0,1}^2(M_a, |z|^2)$ ,  $a < a_0$ .

*Proof.* Following [Na], assume by contradiction that there is a sequence  $\{u_{a_\nu}\} \in L_{0,1}^2(M_{a_\nu}, |z|^2) \cap \text{Dom}(\bar{\partial}_{M_{a_\nu}}) \cap \text{Dom}(\bar{\partial}_{M_{a_\nu}}^*)$ ,  $a_\nu \rightarrow 0$ , such that

$$\|u_{a_\nu}\| = 1, \quad (3.11)$$

whereas

$$\|\bar{\partial}_{M_{a_\nu}} u_{a_\nu}\| + \|\bar{\partial}_{M_{a_\nu}}^* u_{a_\nu}\| < a_\nu. \quad (3.12)$$

We now want to show that  $\{u_{a_\nu}\}$  is a Cauchy sequence.

Remember that  $M_{a_\nu} = M$  outside  $K$ . We now choose a slightly larger compact  $K_1$  containing  $K$  in its interior, and a smooth cut-off function  $\chi$  such that  $\chi \equiv 1$  outside  $K_1$  and  $\chi \equiv 0$  in a neighborhood of  $K$ . Since  $\bar{\partial}_{M_{a_\nu}}$ ,  $\bar{\partial}_{M_{a_\nu}}^*$  coincide with  $\bar{\partial}_M$ ,  $\bar{\partial}_M^*$  outside  $K$ , we obtain from (3.2)

$$2\|\chi u\|^2 \leq \|\bar{\partial}_M(\chi u)\|^2 + \|\bar{\partial}_M^*(\chi u)\|^2$$

for all  $u \in L^2_{0,1}(M_a, |z|^2)$ , which implies

$$\|\chi u\|^2 \leq C'(\|\bar{\partial}_M u\|^2 + \|\bar{\partial}_M^* u\|^2 + \int_{K_1 \setminus K} |u|^2 dV) \quad (3.13)$$

for some constant  $C' > 0$ .

On the other hand, let  $\eta$  be a smooth cut-off function so that  $\eta \equiv 1$  in a neighborhood of  $K_1$ . Then  $\|\eta u_{a_\nu}\|_{\frac{1}{2}}$  is bounded by (3.10), so the generalized Rellich lemma implies that the sequence  $\{u_{a_\nu}\}$  restricted to  $K_1$  is precompact in  $L^2_{0,1}(K_1)$ . Thus it is no loss of generality to assume that the restriction of  $\{u_{a_\nu}\}$  to  $K_1$  is a Cauchy sequence. But this combined with (3.13) implies that  $\{u_{a_\nu}\}$  is a Cauchy sequence in  $L^2_{0,1}(M, |z|^2)$ .

Denote by  $u_0$  the limit of this sequence. From (3.12) it follows that  $\bar{\partial}_M u_0$  and  $\bar{\partial}_M^* u_0$ , defined in the distribution sense, both vanish. But from (3.11) it also follows that  $\|u_0\| = 1$ . This contradicts (3.2) and therefore completes the proof of the lemma.  $\square$

*Proof of theorem 2.1 for  $M$  as above.*

Let  $f$  be given. Then  $\bar{\partial}_{M_a} f$  has compact support and tends to zero when  $a$  tends to zero. It is well known (see e.g. [H2]) that the a priori estimate (3.2) implies that we can solve the equation  $\bar{\partial}_{M_a} u_a = \bar{\partial}_{M_a} f$  with  $\|u_a\| \leq C\|\bar{\partial}_{M_a} f\|$ . Hence  $u_a$  is as small as we wish in  $L^2(M_a, |z|^2)$ , provided  $a$  is small enough. It is well-known that the subelliptic estimate (3.10) implies also the following: Suppose given a compact  $K \subset M_a$  and two smooth real functions  $\zeta, \zeta_1$  with  $\text{supp} \zeta \subset \text{supp} \zeta_1 \subset K$  and  $\zeta_1 = 1$  on  $\text{supp} \zeta$ , then for any integer  $m \in \mathbb{N}$  there exists a constant  $C_{K,m}$  such that

$$\|\zeta u\|_{m+\varepsilon}^2 \leq C_{K,m}(\|\zeta_1 \bar{\partial}_{M_a} u\|_m^2 + \|\zeta_1 \bar{\partial}_{M_a}^* u\|_m^2 + \|\zeta_1 u\|^2)$$

Here  $\|\cdot\|_m$  denotes the Sobolev norm of order  $m$ . But then, choosing the minimal solution satisfying  $\bar{\partial}_{M_a}^* u = 0$ , also the  $\mathcal{C}^\ell$ -norm of  $u_a$  over a given compact  $K \subset M_a$  can be controlled by some  $\mathcal{C}^m$ -norm of  $\bar{\partial}_{M_a} u_a = f$ , and hence made small when letting  $a$  tend to zero. Setting  $f_a = f - u_a$  proves the first statement.

Moreover,  $u_a$  has compact support: Since the  $CR$  structures of  $M$  and  $M_a$  coincide outside a compact set, and  $u_a$  solves the equation  $\bar{\partial}_{M_a} u_a = \bar{\partial}_{M_a} f$ ,  $u_a$  is a  $CR$  function on  $M$  outside some compact set  $K$ . It is no loss of generality to assume that  $M \setminus K$  is connected. But then, since the Hartogs phenomenon for  $CR$  functions holds in 2-pseudoconcave  $CR$  manifolds [LT], the restriction of  $u_a$  to  $M \setminus K$  extends to a  $CR$  function  $\tilde{u}_a$  on  $M$ . Since  $u_a$  belongs to  $L^2_{0,0}(M, |z|^2)$ , the same is true for  $\tilde{u}_a$ . But then (3.3) implies  $\tilde{u}_a \equiv 0$ . Hence  $u_a$  vanishes on  $M \setminus K$ .  $\square$

## 4 The general case

In this section we will explain the proof of Theorem 2.1 for a general 2-pseudoconcave quadratic  $CR$  submanifold  $M$  of type  $(n, d)$  given by

$$M = \{z \in \mathbb{C}^{n+d} \mid \operatorname{Im} z_\ell = \sum_{i,j=1}^n h_{ij}^\ell z_i \bar{z}_j, \ n+1 \leq \ell \leq n+d\}.$$

In this case,  $T^{1,0}M$  is spanned by

$$L_j = \frac{\partial}{\partial z_j} + i \sum_{\ell=n+1}^{n+d} \sum_{k=1}^n h_{jk}^\ell \bar{z}_k \frac{\partial}{\partial x_\ell} \quad j = 1, \dots, n,$$

and  $T^{0,1}M$  is spanned by

$$\bar{L}_j = \frac{\partial}{\partial \bar{z}_j} - i \sum_{\ell=n+1}^{n+d} \sum_{k=1}^n h_{kj}^\ell z_k \frac{\partial}{\partial x_\ell} \quad j = 1, \dots, n.$$

First we show that the analogue of Lemma 3.1 still holds true, i.e. we have the following

### Lemma 4.1

Let  $M$  be a 2-pseudoconcave quadratic  $CR$  submanifold.

1. For all  $u \in L_{0,1}^2(M, |z|^2) \cap \operatorname{Dom}(\bar{\partial}_M) \cap \operatorname{Dom}(\bar{\partial}_M^*)$  we have

$$\|u\|^2 \leq \|\bar{\partial}_M u\|^2 + \|\bar{\partial}_M^* u\|^2. \quad (4.1)$$

2. For all  $u \in L_{0,0}^2(M, |z|^2) \cap \operatorname{Dom}(\bar{\partial}_M)$  we have

$$\|u\|^2 \leq \|\bar{\partial}_M u\|^2 \quad (4.2)$$

*Proof of Lemma 4.1.* We show how the proof of Lemma 3.1 generalizes to this more general setting. In fact, we again use the partial Fourier transform with respect to the variables  $(x_{n+1}, \dots, x_{n+d})$ . For a fixed  $\xi \in \mathbb{R}^d$ , we define the hermitian matrix

$$h^\xi = \sum_{\ell=n+1}^d H_\ell \xi_\ell, \quad \text{i.e. } h_{jk}^\xi = \sum_{\ell=n+1}^d h_{jk}^\ell \xi_\ell.$$

After possibly making a unitary change of coordinates in the variables  $(z_1, \dots, z_n)$ , we may assume that  $h^\xi$  is diagonal with diagonal entries  $h_{jj}^\xi = \lambda_j$  with  $\lambda_1 \leq \dots \leq \lambda_n$ .

Then, as in the proof of Lemma 3.1 we compute  $\widetilde{\bar{\partial}_M u}(z, \xi) = \bar{\partial}_{(z)} \tilde{u}(z, \xi)$  with

$$\bar{\partial}_{(z)} v(z, \xi) = \sum_{k,s=1}^n \bar{\partial}_k v_s d\bar{z}_k \wedge d\bar{z}_s,$$

where

$$\begin{aligned} \bar{\partial}_k v_s &= \frac{\partial}{\partial \bar{z}_k} v_s + \sum_{\ell=n+1}^{n+d} \sum_{m=1}^n h_{mk}^\ell z_m \xi_\ell v_s \\ &= \frac{\partial}{\partial \bar{z}_k} v_s + \sum_{m=1}^n h_{mk}^\xi z_m v_s \\ &= \frac{\partial}{\partial \bar{z}_k} v_s + \lambda_k z_k v_s \end{aligned}$$

Similarly we get

$$\widetilde{\bar{\partial}_M^* u}(z, \xi) = \delta_{(z)} \tilde{u}(z, \xi),$$

where

$$\delta_{(z)} v(z, \xi) = \sum_{j=1}^n (\delta_j v_j)(z, \xi)$$

with

$$\begin{aligned} \delta_j v_j &= -\frac{\partial}{\partial z_j} v_j + \sum_{\ell=n+1}^{n+d} \sum_{k=1}^n h_{jk}^\ell \bar{z}_k \xi_\ell v_j - \bar{z}_j v_j \\ &= -\frac{\partial}{\partial z_j} v_j + \sum_{k=1}^n h_{jk}^\ell \bar{z}_k v_j - \bar{z}_j v_j \\ &= -\frac{\partial}{\partial z_j} v_j + \sum_{k=1}^n \lambda_j \bar{z}_j v_j - \bar{z}_j v_j. \end{aligned}$$

The commutator of  $\bar{\partial}_k$  and  $\delta_j$  can be computed as

$$[\bar{\partial}_k, \delta_j] = (-1 + 2\lambda_j) \delta_{j,k}, \quad (4.3)$$

where  $\delta_{j,k}$  denotes the Kronecker symbol.

Therefore, as in the proof of Lemma 3.1, one obtains for  $v \in \mathcal{D}^{0,1}(\mathbb{C}^n \times \mathbb{R}^d)$ :

$$\begin{aligned} \|\bar{\partial}_{(z)} v\|_z^2 + \|\delta_{(z)} v\|_z^2 &= \sum_{j \neq k} \|\bar{\partial}_j v_k\|_z^2 + \sum_{j=1}^n \|\delta_j v_j\|_z^2 \\ &\geq \sum_{j \neq k} \|\bar{\partial}_j v_k\|_z^2. \end{aligned} \quad (4.4)$$

Now we fix  $k \in \{1, \dots, n\}$ . Since  $M$  is 2-pseudoconcave, the hermitian matrix  $h^\xi$  has at least 2 negative and 2 positive eigenvalues. But this implies that there exist indices  $r, s \neq k$  such that  $\lambda_r < 0$  and  $\lambda_s > 0$ . We now define real numbers  $a_j \in [0, 1]$  by

$$\begin{aligned} a_j &= 0, \quad j \neq r, s \\ a_r &= \frac{\lambda_s}{\lambda_s - \lambda_r} \\ a_s &= \frac{-\lambda_r}{\lambda_s - \lambda_r} \end{aligned}$$

Note that by definition of  $a_j$  we have  $\sum_{j=1}^n a_j = 1$  and  $\sum_{j=1}^n a_j \lambda_j = 0$ . But then, using (4.3) we obtain

$$\begin{aligned} \sum_{\substack{j=1 \\ j \neq k}}^n \|\bar{\partial}_j v_k\|_z^2 &\geq \sum_{\substack{j=1 \\ j \neq k}}^n a_j \|\bar{\partial}_j v_k\|_z^2 \\ &= \sum_{j=1}^n a_j \|\delta_j v_k\|_z^2 + \sum_{j=1}^n (1 - 2\lambda_j) a_j \|v_k\|_z^2 \\ &\geq \sum_{j=1}^n a_j \|v_k\|_z^2 - 2 \sum_{j=1}^n \lambda_j a_j \|v_k\|_z^2 \\ &\geq \|v_k\|_z^2. \end{aligned}$$

From (4.4) we therefore obtain

$$\|\bar{\partial}_{(z)} v\|_z^2 + \|\delta_{(z)} v\|_z^2 \geq \|v\|_z^2.$$

By reasoning as in the proof of Lemma 3.1 we may therefore conclude that (4.1) holds.

Likewise, for the proof of (4.2), we define real numbers  $c_j \in [0, 1]$  by

$$\begin{aligned} c_j &= 0, \quad j \neq 1, n \\ c_1 &= \frac{\lambda_n}{\lambda_n - \lambda_1} \\ c_n &= \frac{-\lambda_1}{\lambda_n - \lambda_1} \end{aligned}$$

Then we have  $\sum_{j=1}^n c_j = 1$  and  $\sum_{j=1}^n c_j \lambda_j = 0$ . Therefore (4.3) implies

$$\begin{aligned}
\sum_{j=1}^n \|\bar{\partial}_j v\|_z^2 &\geq \sum_{j=1}^n c_j \|\bar{\partial}_j v\|_z^2 \\
&= \sum_{j=1}^n c_j \|\delta_j v\|_z^2 + \sum_{j=1}^n c_j (1 - 2\lambda_j) \|v\|_z^2 \\
&\geq \sum_{j=1}^n c_j \|v\|_z^2 - 2 \sum_{j=1}^n c_j \lambda_j \|v\|_z^2 \\
&= \|v\|_z^2,
\end{aligned}$$

This completes the proof of (4.2) by the same arguments as in the proof of Lemma 3.1.  $\square$

*Remark:* The proof of this Lemma is essentially contained in [Na] with constants depending on the Levi form of  $M$ . Here we have shown that one can take the same constant 1 for every 2-pseudoconcave quadratic  $CR$  submanifold  $M$ .

The second essential ingredient for the proof of Theorem 2.1 in the general case is the subelliptic estimate proved for 2-pseudoconcave  $CR$  manifolds of arbitrary codimension  $d$  in [HN]: There exists  $\varepsilon > 0$  such that for every compact  $K$  of  $M$ , there exists a constant  $C_K > 0$  independent of  $a$  such that

$$\|u\|_\varepsilon^2 \leq C_K (\|\bar{\partial}_{M_a} u\|^2 + \|\bar{\partial}_{M_a}^* u\|^2 + \|u\|^2) \quad (4.5)$$

for all  $u \in \mathcal{D}_K^{0,1}(M_a)$ . This subelliptic estimate replaces (3.10) in the general situation.

Using (4.1) and (4.5), one can prove the uniform  $L^2$  a priori estimate for  $\bar{\partial}_{M_a}$  as stated in Lemma 3.2. The proof is the same. But this, together with (4.2) completes the proof of Theorem 2.1 as in section 3.

## References

- [FK] G.B. FOLLAND, J.J. KOHN: *The Neumann problem for the Cauchy-Riemann complex*. Ann. Math. Studies **75**, Princeton University Press, Princeton, N. J. (1972).
- [H1] L. HÖRMANDER:  *$L^2$  estimates and existence theorems for the  $\bar{\partial}$  operator*. Acta Math. **113**, 89–152 (1965).
- [H2] L. HÖRMANDER: *An introduction to complex analysis in several complex variables*. North Holland Mathematical Library (1990).

- [HN] C.D. HILL, M. NACINOVICH: *Pseudoconcave CR manifolds*. Preprint, Dipartimento de matematica, Pisa 1-76, 723 (1993). In: Complex analysis and geometry (V. Ancona, E. Ballico, A. Silva, eds), Lecture notes in pure and applied mathematics vol. **173**, Marcel Dekker, New York, 275–297 (1996).
- [JT] H. JACOBOWITZ, F. TRÈVES: *Non-realizable CR structures*, Invent. Math. **66**, 231–249 (1982).
- [LT] CH. LAURENT-THIÉBAUT: *Résolution du  $\bar{\partial}_b$  à support compact et phénomène de Hartogs-Bochner dans les variétés CR*. Proc. Sympos. Pure Math. **52**, 239–249 (1991).
- [Na] I. NARUKI: *Localization principle for differential complexes and its applications*. Publ. RIMS **8**, 43–110 (1972).
- [Ni] L. NIRENBERG: *On a problem of Hans Lewy*. Uspeki Math. Naut. **292**, 241–251 (1974).
- [R] H. ROSSI: *Attaching analytic spaces to an analytic space along a pseudoconcave boundary*, Proc. Conf. Complex Manifolds (Minneapolis), 1964, Springer-Verlag, New York, 242–256 (1965).